Finite Simple Groups

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1 Introduction

The motivation for this project to understand finite simple groups which reveal more information about all finite groups. Just like primes in number theory and atoms in chemistry, finite simple groups are the building blocks out of which all finite groups are made.

Comprehensive classification of finite simple groups is attributed to Daniel Gorenstein in 1981, but it was not completed until revisions were made by Michael Aschbacher and Stephen Smith to correct some errors in the proof [5].

2 Definitions and Classification

Recall that a subgroup H of a group G is called a *normal subgroup* of G if $aH = Ha$ for all a in G and is denoted $H \triangleleft G$. For every group G, the subgroups $\{e_G\}$ (the identity subgroup) and G are normal in G. We say that a subgroup H is *proper* if $H \leq G$ and $H \neq G$.

Definition 2.1 (Simple Group [6])**.** *A group is simple if its only normal subgroups are the identity subgroup and the group itself.*

In other words, none of the proper nontrivial subgroups of a simple group are normal. A natural question then arises: is the group of order 1 simple? The group of order 1 contains only the identity element. Thus, this group has no proper nontrivial subgroups, so there are certainly no normal subgroups. Hence, the group of order 1 is not simple because by definition, a simple group must have exactly two normal subgroups.

2.1 Jordan-Hölder Theorem

Finite simple groups can be considered the building blocks for all finite groups, and they can be determined in the following way: given a finite group $G = G_0$, choose a proper normal subgroup $G_1 \leq G = G_0$ of largest order. Then the factor group G_0/G_1 is simple. Then we choose a proper normal subgroup $G_2 \leq G_1$ of largest order. Then G_1/G_2 is simple and so on. The result is then $G_n = \{e\}$.

Definition 2.2 (Composition Series and Factors [2]). *Every finite group G with order greater than one has a finite series of subgroups called a composition series such that*

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G$$

where H_{i+1} is a maximal subgroup of H_i and $H \lhd G$. Additionally, a composition factor H_{i+1}/H is a simple group for $0 \leq i \leq k-1$.

A composition series can be thought of as a series where it is impossible to add a new member which is distinct from all other members. The following theorem expands on this idea.

Theorem 2.1 (Jordan-Hölder). *Let G be a nontrivial finite group. Then the composition factor belonging to two composition series of G are isomorphic in pairs. In other words, let*

$$G = K_0 \lhd K_1 \lhd \cdots \lhd K_n = \{e\}$$
$$G = H_0 \lhd H_1 \lhd \cdots \lhd H_m = \{e\}$$

be any two composition series for G. Then $n = m$, and corresponding to any composition factor K_j/K_{j+1}, there is a composition factor H_i/H_{i+1} such that

$$\frac{K_j}{K_{j+1}} \cong \frac{H_i}{H_{i+1}}.$$

Sketch of proof. We use induction over the length of shortest composition series for G. It is sufficient to show that any composition series is equivalent to a minimal series and thus any two series are equivalent. For the base case, notice that if G is simple, then it has a unique composition series $G \lhd \{e\}$ and we are done. Also notice that if $|G| = 2$, then the theorem is trivially true. Then for the inductive step, we define two composition series

$$G = K_0 \lhd K_1 \lhd \cdots \lhd K_n = \{e\}$$
$$G = H_0 \lhd H_1 \lhd \cdots \lhd H_m = \{e\}$$

and consider two cases. If $K = H$, then it directly follows that the composition factors are isomorphic in pairs and the theorem is true. For the case where $K \neq H$, the proof becomes more involved. We consider the group KH which contains K and H distinct and maximal in G such that $KH = G$. Utilizing the first isomorphism theorem and the fact that the composition factors are simple, we have that $K \cap H$ is a maximum normal subgroup in K and H. Then using the inductive assumption, the theorem is true for K and H, so the composition factors are isomorphic in pairs and the theorem is true for $K \neq H$ as well. □

In terms of composition factors, this theorem states that these factors are independent of the normal subgroups chosen. In addition, a group can be reconstructed from its composition factors and many group properties are determined by its composition factors.

2.2 Relation to Sylow Theory

We now use Sylow theory to draw some more conclusions about simple groups. First, we give the relevant Sylow Theorem and its consequences.

Theorem 2.2 (Sylow's Third Theorem [6])**.** *Let p be a prime and let G be a group of order $p^k m$, where p does not divide m. Then the number n of Sylow p-subgroups of G is equal to $1 \mod p$ and divides m.*

Corollary 2.2.1. *A Sylow p-subgroup is normal in G if and only if it is the unique Sylow p-subgroup.*

We define n_p as the number of Sylow p-subgroups. Thus Corollary 2.2.1 is equivalent to stating that $n_p = 1$.

Theorem 2.3 (Sylow Test for Nonsimplicity [6])**.** *Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is equal to $1 \mod p$, then there does not exist a simple group of order n.*

Sylow's Third Theorem can be used to establish statements such as: *there are no simple groups of order k (for some k).* To do this, we simply need to show that $n_p = 1$ for some p dividing the order of G. Thus from Corollary 2.2.1, the Sylow p-subgroup is normal and the group has another normal subgroup besides itself and the trivial group.

Example 2.1. There are no simple groups of order 84.
Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the Third Sylow Theorem tells us that n_7 divides $2^2 \cdot 3 = 12$ so $n_7 \in \{1, 2, 3, 4, 6, 12\}$. Additionally, we have that $n_7 \equiv 1 \mod 7$. Thus the only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. Hence, any group of 84 cannot be simple by definition.

Example 2.2. There are no simple groups of order 351.
Since $|G| = 351 = 3^3 \cdot 13$, Sylow's Third Theorem tells us that n_{13} divides $3^3 = 27$ so $n_{13} \in \{1, 3, 9, 27\}$. Additionally, we have that $n_{13} \equiv 1 \mod 13$. Thus the only possibilities are $n_{13} = 1$ or 27. A Sylow 13-subgroup P has order 13 and a Sylow 3-subgroup Q has order $3^3 = 27$. Thus $P \cap Q = \{e\}$. Suppose $n_{13} = 27$. Every Sylow 13-subgroup contains 12 nonidentity elements, so G must contain $27 \cdot 12 = 324$ elements of order 13. This leaves $351 - 324 = 27$ elements in G with order not 13. Thus, G contains only one Sylow 3-subgroup, so G cannot be simple.

2.3 Classification

To broaden our understanding of finite simple groups, it is helpful not only to explain why a group cannot be simple but also to see which groups classify as simple. The classification of finite simple groups provides a complete list of all finite simple groups, but contrary to the label of "simple groups," there are infinitely many distinct finite simple groups.

Theorem 2.4 (Classification of Finite Simple Groups)**.** *Every finite simple group is (isomorphic to) one of the following:*

1. *Cyclic group $\mathbb{Z}_p$ of prime order p*

2. *Alternating group A_n for $n \geq 5$*
3. *One of the 16 infinite families of groups of Lie Type*
4. *One of the 26 sporadic groups*

3 Cyclic Groups of Prime Order

As a straightforward example, consider the cyclic group $\mathbb{Z}_p$, which are simply the integers mod p with p prime.

Example 3.1. Consider the cyclic group $G = (\mathbb{Z}_3, +)$, the group of integers under addition modulo 3. To show that G is a simple group, we must show that its only normal subgroups are the identity subgroup and the group itself. If H is a subgroup of G, then its order must divide $|G| = 3$. Since $p = 3$ is prime, its only divisors are 1 and 3, so either $H = G$ or H is the trivial (identity) group.

From this example, we can see that since p is prime, its only divisors are 1 and p. Thus, from Legrange's Theorem, a subgroup must either be the whole group or the trivial group. This hints at the fact that primality is key to classifying abelian simple groups.

Theorem 3.1. *Every abelian simple group is (isomorphic to) $\mathbb{Z}_p$.*

Proof. ($\Longrightarrow$) Let G be an abelian simple group. If $a \in G$ is a non-identity element, then $\langle a \rangle$ is a nontrivial subgroup of G. Any subgroup of an abelian group is normal, so since G is abelian, $\langle a \rangle$ is a nontrivial normal subgroup of G. Then since G is also simple, we must have that $G = \langle a \rangle$, or equivalently, G is cyclic. If G is infinite, then $G \cong \mathbb{Z}$. $\mathbb{Z}$ contains proper nontrivial subgroups (i.e., $\langle a^2 \rangle \leq \langle a \rangle = G$) so it is not simple. Then G must be a finite cyclic group. If $G \cong \mathbb{Z}_n$ where $n = ab$ for $a \neq 1, b \neq 1$, then G contains a proper subgroup of order a. If $G \cong Z_p$ where p is prime, then by Legrange's Theorem, the only subgroups are $\langle 0 \rangle$ and G, which implies that G is simple. Thus all abelian simple groups are of the form Z_p for p prime, up to isomorphism.
($\Longleftarrow$) Suppose G is a group of prime order p and let $a \in G$ be a nonidentity element. Then $\langle a \rangle$ divides $|G|$ and $|\langle a \rangle| = p$ since p is prime. Thus $G = \langle a \rangle$ so G is cyclic and abelian. Since any normal subgroup of $H \leq G$ must have either order 1 or p, H must either be trivial or all of G. Hence G is a simple abelian group, by definition. □

While the cyclic group of prime order are straightforward to understand, non-abelian simple groups are far more complicated.

4 Alternating Groups

The Alternating group of degree n is defined as "The group of even permutations of n symbols is denoted by A_n." in the textbook. Logically, they are the subgroup of the Symmetry group S_n that consists all of the even permutations of S_n itself. Interestingly, all Alternating Groups, A_n, are finite simple groups, with the exception that $n = 4$. Here will we show that A_4 is not simple and how A_n, $n \geq 5$, is simple.

Theorem 4.1. *Alternating group A_4 is not simple*

Proof. We consider the subgroup $H = [1, (12)(34), (13)(24), (14)(23)]$ in A_4. We can verify that this is a normal subgroup by knowing that conjugation ($gHg^{-1} = H$) does not alter cycle structure. Outside of the identity element, all of the elements are products of two disjoint transpositions and therefore must equal another product of two disjoint transpositions. Therefore H is normal and A_4 cannot be simple. □

We can see H is normal by observing that it is isomorphic to the Klein 4-Group, $\mathbb{Z}_2 \times \mathbb{Z}_2$

Theorem 4.2. *Every Alternating group of order $n \geq 5$ is simple*

Proof. We break up the proof into the following steps, as originally presented by Gregory Constantine in [3]:

Step 1. A_n is generated by the 3-cycles. In fact A_n is generated by the $n-2$ 3-cycles $\{(12k) : k \geq 3\}$.

Indeed, any element of A_n is a product of transpositions of the form $(ab)(cd)$ or $(ab)(ac)$. Since $(ab)(cd) = (acb)(acd)$ and $(ab)(ac) = (acb)$ we conclude that A_n is generated by the 3-cycles. Furthermore, $(1a2) = (12a) - 1, (1ab) = (12b)(12a) - 1, (2ab) = (12b) - 1(12a)$, and $(abc) = (12a) - 1(12c)(12b) - 1(12a)$, which shows that every 3-cycle is generated by a cycle of the form $(12k)$.

Step 2. If H is a normal subgroup of A_n and H contains a 3-cycle, then $H = A_n$.

Without loss $(123) \in H$. Then

$$(12k) = ((12)(3k))(123)^{-1}((12)(3k))^{-1} = ((123)^{-1})^{(12)(3k)} \in H,$$

by normality. Thus $A_n =< (12k)$: all $k \geq 3 \leq H$, and $H = A_n$.

We now assume that $H(\neq 1)$ is normal in A_n. The idea is to show that H contains a 3-cycle, necessarily. We then invoke Step 2 to conclude the proof. Consider the exhausting possibilities examined below:

Case 1. Without loss assume that H contains $\sigma = (12 \cdots r)\tau$, where $r \geq 4$, and τ is disjoint of $\{1, 2, \ldots, r\}$. Then, by the normality of H, H contains $\sigma^{-1}\sigma^{(123)} = (12r)$. We are done by Step 2.

Case 2. Assume without loss that $\sigma = (123)(456)\tau \in H$, where τ is a product of disjoint transpositions. Then H contains $\sigma^{-1}\sigma^{(124)} = (14263)$. We are done by **Step 1**.

Case 3. Assume without loss that $(123)\tau \in H$, with τ a product of disjoint transpositions. Then H contains $(123)\tau(123)\tau = (132)$, and we are done by **Step 2.**

Case 4. Assume that H contains elements that are products of disjoint transpositions. Without loss let $(12)(34)\tau$ be such an element of H.

Then $(12)(34)\tau((12)(34)\tau)^{(123)} = (13)(24) \in H$. Since $n \geq 5$, consider $(123) \in A_n$. By normality H contains $(13)(24)((13)(24))(135) = (135)$, and we are done by **Step 2**. This concludes the proof.

□

This is a complex proof. In simple terms, it can be broken up as following: Determine that for $n \geq 5$, all elements of A_n are products of 3-cycles. All 3-cycles in A_n are conjugate, and therefore are normal. For any A_n, this 3-cycle is the only nontrivial normal subgroup. Thus, A_n is equal to the 3-cycle. A_n must then be a finite simple group.

Example 4.1. Show A_6 is finite simple

We set H to be the set of conjugacy classes,

$$H = [(1), (123), (123)(456), (12)(34), (12345), (23456), (1234)(56)].$$

(123) is the only 3-cycle element that is invariant under conjugation. The only normal subgroups of A6 are (1) and elements in the form (123), so A_6 is a simple finite group.

5 Sporadic Groups

5.1 Sporadic Groups

There are 26 finite simple groups that do not fit into any of the four infinite families of finite simple groups (i.e., the cyclic groups of prime order, alternating groups of degree at least five, Lie-type Chevalley groups, and Lie-type groups) which compose sporadic simple groups. We provide a summary of these groups in table 1.

All of the 26 sporadic groups mentioned in table 1 have been proven to be a complete list of all the sporadic groups which can be used to describe various groups in mathematics. In fact, all the finite simple groups were classified in a groundbreaking proof presented as the Enormous Theorem in 2004 that took decades and hundreds of minds to complete.

5.2 Sporadic Group Families

Despite being sporadic, all the sporadic groups can be classified into 3 major families with 6 being the outliers. The classifications can be analyzed using the centralizers of the groups or automorphisms in relation to the Monster group. Below is a detailed representation of sporadic group families:

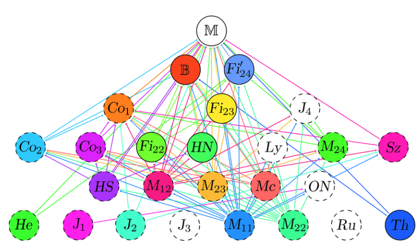


Figure 1: Families of sporadic groups.

Table 1: Order of all groups belonging to the set of finite simple sporadic groups.

Name	Order
Mathieu group M_{11}	7920
Mathieu group M_{12}	95040
Janko group J_1	175560
Mathieu group M_{22}	443520
Janko group $J_2 = HJ$	604800
Mathieu group M_{23}	10200960
Higman-Sims group HS	44352000
Janko group J_3	50232960
Mathie group M_{24}	244823040
McLaughlin group McL	898128000
Held group He	4030387200
Rudvalis group Ru	145926144000
Suzuki group Suz	448345497600
O'Nan group $O'N$	460815505920
Conway group Co_3	495766656000
Conway group Co_2	42305421312000
Fischer group Fi_{22}	64561751654400
Harada-Norton group HN	273030912000000
Lyons group Ly	51765179004000000
Thompson group Th	90745943887872000
Fischer group Fi_{23}	4089470473293004800
Conway group Co_1	4157776806543360000
Janko group J_4	86775571046077562880
Fischer group Fi'_{24}	1255205709190661721292800
Baby monster group B	4154781481226426191177580544000000
Monster group M	808017424794512875886459904961710757005754368000000000

5.3 Monster Group - The largest sporadic group

The Monster group is the largest of all the sporadic groups with order of

$$808017424794512875886459904961710757005754368000000000.$$

The group was originally predicted to exist in the early 1970s by B.Fischer and R.L. Griess. The group describes symmetries of a 196883 dimensional object and each element of the Monster group takes up about 4GB of computational space. In fact, R.A.Wilson supplied two matrices that generate the monster group with each of the matrices taking up about 5GB of space. Notice that there are many groups a lot larger than the monster group, such as permutations of 101 objects, but their elements can be computed in just a few seconds.
The monster group can be understood using the following statements:

- It is the largest sporadic simple group or alternatively, the unique simple group of its order.
- It is the automorphism group of the Griess algebra.
- It is the automorphism group of the monster vertex algebra.
- It is a group of diagram automorphisms of the monster Lie algebra.

5.4 Sporadic groups outside of math

Despite the fact that we have not fully discovered applications of the monster group or many other sporadic groups, the order of the monster group has appeared in other fields. The first-ever appearance of the order of monster group appeared in In fact, the monstrous moonshine theory appeared as a result of an unexpected connection between the monster group and modular functions. It was proved by Richard Borcherds in 1998 for which he received the Fields Medal

5.5 Summary

While there are many sporadic groups, the most fascinating thing about them is the fact that every single one has been proven to exist. Moreover, the potential applications of these groups can shine a light on the connections between math and other sciences.

6 Groups of Lie Type

6.1 Lie groups and Lie algebras

Up to this point, groups have primarily satisfied a discrete representation. Whether or not a group has been finite or infinite, it has typically appeared to be somewhat countable set. For example, cyclic groups of prime order p can be described as a finite set of integers mod p and the subgroup of triangle rotations in D_3 occupy regular intervals of $120°$.

But what prevents groups from being (in some imprecise sense) *granular*? Take for example the group of rotations of an equilateral triangles. Is the set of rotations still a group if we take rotations by arbitrarily small degree increments? Clearly not; purely from geometric intuition, we see that the triangle no longer maintains symmetry. But what if we observe a circle instead of a triangle? Suddenly, we find that its rotations do appear to form a group, and in fact is true.

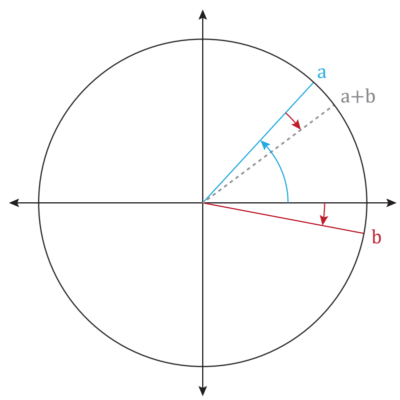


Figure 2: Circle group $\mathbb{T}$, the additive group of rotations of a circle. This is a trivial illustrative example of a Lie group.

Typically, the group of rotations of the unit circle is denoted by $\mathbb{T}$. As we've suggested, the introduction of arbitrarily small degree increments makes this group a bit different from the groups we've seen previously. Beyond just having an infinite order, we can observe that this group (purely geometrically) satisfies some interesting form of *continuous symmetry*; in some sense, the group seems to be *continuous*.

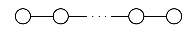
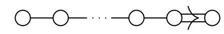
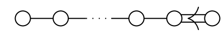
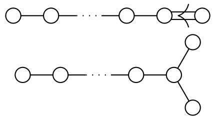
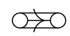
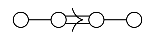
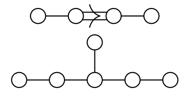
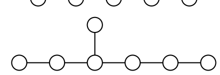
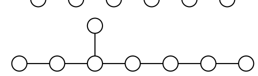
As we find, these concepts of continuous symmetries and continuous groups—known as *Lie groups*—actually give way to rich theory that intimately connects abstract algebra and topology. While we advise the curious reader to seek out a more formal and precise introduction to this fascinating corner of mathematics, for the purpose of this paper we will take a sufficiently abstract view. Our goal here is to provide a sliver of intuition behind why these objects are fundamental to group theory and how (perhaps paradoxically) their continuity gives way to a rich class of *finite* simple groups.

Definition 6.1 (Lie group [4])**.** *A Lie group is a group that is also a finite dimensional smooth manifold, in which the group operations of multiplication and inversion are smooth maps.*

Shortly after their discovery in the late 19th century by Sophus Lie, Lie groups ushered in a split in group theoretic research. Group theorists focused on discrete groups (of finite/infinite order) and continuous/Lie groups separately, but quickly found that studying Lie groups could help towards understanding finite groups as well.

Just like their finite counterparts, Lie groups also give way to simple groups of the same type—for this paper, it is not critical to understand the technicalities behind their definition. Just like finite simple groups, classifying simple Lie groups was its own technical challenge and of independent interest. Following the classification of simple Lie

Table 2: Classification of simple Lie groups and their corresponding Dynkin diagrams.

Classical		
A_n	Group of $n \times n$ *unitary* matrices with determinant 1 (Special unitary Group $A_n = SU(n+1)$)	
B_n	Group of $n \times n$ rotation matrices with determinant 1 (odd special orthogonal groups $B_n = SO(2n+1)$).	
C_n	(unitary simplectic groups)	
D_n	(even special orthogonal groups)	
Exceptional		
G_2		
F_4		
E_6		
E_7		
E_8		

groups by Killing in [9], group theorists began thinking about how their discovery could be bridged to the world of finite groups.

Theorem 6.1 (Classification of simple Lie groups [9])**.** *Every simple Lie group can be classified as one of the four* classical *Lie groups (A_n, B_n, C_n, D_n) or one of five* exceptional *Lie groups (G_2, F_4, E_6, E_7, E_8).*

While the classical simple Lie groups had been known for centuries, the discovery of 5 new exceptional groups by Killing ultimately led to the full classification. Although we will not go into depth into the methods of this classification, we point to table 2 and appendix A for insight into these groups' structure. As we lack succinct geometric interpretations, in appendix A we additionally describe how Dynkin Diagrams can help us describe groups of Lie type (and Lie groups in general). Ultimately, the importance of this classification arose from the discovery that these simple continuous groups gave way to analogous finite simple groups.

6.2 Adjoint/untwisted Chevalley groups

In 1955, Chavelley's seminal work [1] revealed the unexpected connection between continuous Lie groups and finite simple groups. Specifically, Chavelley showed that the simple Lie groups classified over a half-century prior had finite analogues. In almost all cases, he found that simple Lie groups defined over finite fields would give way to another (now finite) simple group. Since these groups were not quite Lie groups (by consequence of their discreteness), they came to introduce the notion of a Lie-type group—frequently constructed from a Lie group taken over a finite field. The specific subclass of finite simple groups originally described by Chavalley have come to be known as *adjoint Chavelley groups* and would inspire further techniques and discoveries of other Lie-type finite simple groups.

6.3 Adjoint twisted Chevalley groups

Following Chavalley's discovery, many group theorists started looking to Lie groups and Lie algebras as a means of finding new finite simple groups.

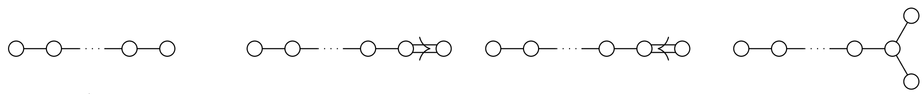
In 1959, five years since Chavalley's paper, Robert Steinberg [11] modified Chavalley's techniques to cover a broader class of known finite simple groups and introduce a few new finite simple groups. Namely, the groups ${}^2A_n, {}^2D_n, {}^2E_6$, and ${}^4D_4$. These came to be known as the *Steinberg Groups*. From 1960-61, Michio Suzuki [12] and Rimhak Ree [10] further modified these techniques to describe more finite simple groups: ${}^2B_2, {}^2F_4$, and ${}^2G_2$. The series of groups associated with ${}^2F_4$ have an exception—the first group in this series is not quite simple (denoted by ${}^2F(2)$). Jacques Tits studied the properties of this group and found that its commutator subgroup was in fact simple. Denoted by ${}^2F(2)'$, this group would be known as the Tits group and is seen as a sort of "exceptional Lie-type" group, in some sense.

6.4 Representations of finite simple groups of Lie type

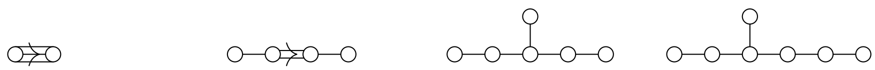
Although almost all finite simple groups of Lie type do not have a simple (geometric) interpretation, group theorists have found different ways of representing these objects. Although for the purpose of this paper we will not go into depth about these representations, we present a full classification in table 3. We further explain the chosen representation (Dynkin diagrams) in appendix A.

Table 3: Classification of finite simple groups of Lie type. Although there is no unified notation, we denote some of these groups equivalently to their underlying Lie groups.

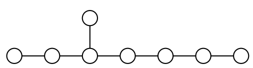
Adjoint Chevalley groups [1]



A_n B_n C_n D_n



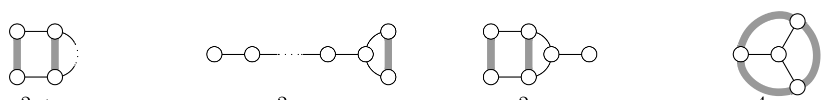
G_2 F_4 E_6 E_7



E_8

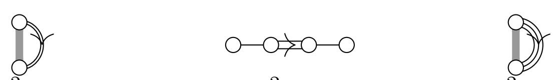
Twisted adjoint Chevalley groups

Steinberg [11]



^2A_n ^2D_n $^2E_6$ $^4D_4$

Suzuki-Ree [12, 10]



$^2B_2$ $^2F_4$ $^2G_2$

Tits

$^2F(2)'$

7 Conclusion

Finite simple groups are an interesting class of groups in which all finite groups are comprised of. Many questions about finite groups can be reduced to questions about simple groups, so determining and understanding all finite simple groups is crucial to group theory. As a result, the complete classification of finite simple groups as mentioned in Theorem 2.4 was a major breakthrough in group theory. In the realm of abelian groups, Theorem 3.1 is interesting since we can classify all abelian simple groups being isomorphic to $\mathbb{Z}_p$ where p is prime.

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Appendices

A Simple groups of Lie type

A.0.1 Lie algebras and Dynkin diagrams

We take a brief detour to discuss how we can visualize the structure of finite simple groups of Lie type. In the world of Lie groups, we typically run across the notion of a *Lie algebra*. While it is not critical to fully understand how these objects behave, we can picture a Lie algebra as follows: if we imagine a Lie group as the differentiable manifold it describes, its associated Lie algebra will be the tangent space at the identity of the Lie group. In the case of $\mathbb{T}$, we can see that this is the tangent line at point $(1, 0)$ on the complex plane (see fig. 2).

It is customary to use Fraktur characters like $\mathfrak{g}$ and $\mathfrak{h}$ to denote Lie algebras.

Definition A.1 (Lie algebra [8, 7])**.** *A* **Lie algebra** *is a vector space* $\mathfrak{g}$ *over a field* F *with an operation* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *which we call a* **Lie bracket**, *such that the following properties are satisfied:*

1. *It is* **bilinear**, *implying*

$$[ax + by, z] = a\,[x, z] + b\,[y, z]$$

$$[z, ax + by] = a\,[z, x] + b\,[z, y]$$

for all scalars $a, b \in F$ *and all* $x, y, z \in \mathfrak{g}$.

2. *It is* **skew symmetric**, *implying* $[x, x] = 0$ *and equivalently* $[x, y] = -\,[y, x]$ *for all* $x, y \in \mathfrak{g}$.

3. *The* **Jacobi identity** *holds:*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$.

Importantly, any Lie group we observe will have an associated Lie algebra. Given that this Lie algebra is built upon purely linear operations, it is often more convenient to work with than the Lie group itself. While there exists many different ways to represent these algebras, a common approach is using *root systems*. Although we will not be going over root systems here, they give way to a finite-graph representation of the Lie algebra through a *Dynkin diagram*—a type of graph with singled/doubled/tripled edges and sometimes is directed (the definition of such a graph is somewhat ambiguous in the literature.

Although Dynkin diagrams are far from the most descriptive representation of Lie structures, they remain as easily-identifiable 'fingerprints' for Lie algebras. In the case of finite simple groups of Lie type, they appear as the identifiers of the Lie algebra associated with the underling Lie group.